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The central two-point connection problem for the reduced confluent Heun equation

A Ya Kazakov

St-Petersburg State University of Aerospace Instrumentation, 67 B Morskaya St, St-Petersburg 190000, Russia

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Abstract

Integral symmetry of the confluent Heun equation reduces the central two-point connection problem for the reduced confluent Heun equation to the asymptotical problem. Some heuristic formulae for the transition matrix are obtained and compared with results of the computational calculations.

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1. Introduction

The analytical theory of the linear ordinary differential equations (LODE) and, specifically, the theory of the special functions is an important part of mathematical and theoretical physics. The central objects of this theory are the monodromy groups that describe the behaviour of the solutions of the LODE for their analytical continuation on the complex plane. Besides, the analytical theory of differential equations contains some other objects describing the global properties of the solutions, namely, the transition matrices [4]. They connect different pairs of solutions (we mean here the second-order differential equations) determined by the behaviour in the vicinities of the singularities. If the set of transition matrices for the given equation is known, one can calculate the monodromy matrices for this equation. The search for the transition matrix is known as a central two-point connection problem [4]. This problem is a natural generalization of the singular Sturm–Liuville problem. If one of the entries of the transition matrix is equal to zero, then it could relate to the solution of the corresponding singular Sturm–Liuville problem.

The well-known 21st Hilbert problem (or direct Riemann–Hilbert problem) constitutes the following question: can a given group be realized as a monodromy group of the equation of a prescribed type? This problem was intensively discussed during recent decades, see [1]. But from the practical point of view the inverse Riemann–Hilbert problem—to calculate the monodromy group for the given LODE, is more important. While the direct Riemann–Hilbert problem is, in essence, the question of uniqueness and existence, the inverse Riemann–Hilbert problem requires the explicit description of a monodromy group. The first achievements in this direction belong to Gauss, Riemann and Kummer, who evaluated the transition matrices (and, respectively, monodromy group) for the hypergeometric equation

$$\frac{\mathrm{d}^2 w(s)}{\mathrm{d}s^2} + \frac{(a+b+1)s - c}{s^2 - s}\frac{\mathrm{d}w(s)}{\mathrm{d}s} + \frac{ab}{s^2 - s} w(s) = 0. \tag{1}$$

Thus, the description of the monodromy groups and Stokes parameters for the LODE of the hypergeometric class (this class includes both the hypergeometric equation and all its confluent and reduced forms) was obtained [2, 3]. Note that the solutions of the hypergeometric class equations can be expressed as contour integrals [2], and it is precisely this fact that gives the possibility of calculating the corresponding monodromy matrices.

The hypergeometric equation (1) is a Fuchsian differential equation with three regular singularities, namely, $z_0 = 0$, $z_1 = 1$, $z_\infty = \infty$. The next in complexity is the Heun equation with four regular singularities, and its confluent and reduced forms (which give the Heun class) [4]. Contrary to the hypergeometric class the solutions of the Heun class equations do not have integral representations and the transition matrices for these equations are not known. In order to obtain an analytical information about the monodromy groups it was suggested in [5] to use the integral relations for the solutions of the different LODE instead of the integral representations. Specifically, the calculation of the monodromy matrices for the confluent Heun equation (CHE)

$$\mathbf{R}_z(\alpha, \beta, \gamma, \delta, \epsilon)v(z) = (z^2 - z)v''(z) + (\alpha z^2 + \beta z + \gamma)v'(z) + (\delta z + \epsilon)v(z) = 0 \tag{2}$$

was reduced to the asymptotic problem for the third-order LODE. Unfortunately, the effective asymptotic technique for the third-order LODE is not known, so the problem of calculation of the transition matrices for CHE has not been solved.

Some additional analysis, however, shows that the integral symmetry gives another possibility in this context, reducing the analytical problem of the calculation of the transition matrices to the asymptotical problem for the second-order LODE. Here we discuss these considerations for the reduced confluent Heun equation (RCHE)

$$(z^2 - z)v''(z) + (\beta z + \gamma)v'(z) + (\delta z + \epsilon)v(z) = 0. \tag{3}$$

This equation has two regular singularities (z_0 and z_1) and an irregular singularity z_∞. We show that the integral symmetry for the CHE reduces the calculation of the transition matrices for the RCHE to the asymptotic problem for the second-order LODE. (Note that equation (3) can be obtained from (2) by the passage to the limit $\alpha \to 0$, and so the large parameter in this asymptotic procedure will be α^{-1}.) Here we need the specific asymptotic technique for the second-order LODE, namely the Cherry version of the comparison equation method, see [6–8]. We derive on this basis some heuristic formulae for the transition matrices for the RCHE, see relation (30). Our calculation has the formal character, and we compare our formula with the results of the computer calculations.

The layout of this paper is as follows. In the next section, we introduce the main analytical objects for the CHE and describe the set of the elementary symmetries for this equation. In section 3, we discuss the integral symmetry for the CHE. These results were obtained in [9] and here we clarify them and make some corrections. Afterwards in section 4, we apply the Cherry version of the comparison equation method in order to derive the transition matrix for the RCHE. In section 5, we compare this result with direct computer calculations.

2. CHE and its elementary symmetries

According to the analytical theory of LODE (see [10, 11]), equation (2) has the following pairs of fundamental solutions fixed by its behaviour in the vicinities of z_0, z_1,

$$V^{(0)}(z) = \left[v_b^{(0)}(z), v_h^{(0)}(z)\right]^T, \qquad V^{(1)}(z) = \left[v_b^{(1)}(z), v_h^{(1)}(z)\right]^T,$$

where $v_b^{(0)}(z) = z^{\rho_0} v^{(0)}(z)$, $v_b^{(1)}(z) = (1-z)^{\rho_1} v^{(1)}(z)$, $\rho_0 = 1+\gamma$, $\rho_1 = 1-\alpha-\beta-\gamma$ are the characteristic exponents of the singularities, functions $v^{(k)}(z)$, $v_h^{(k)}(z)$ are holomorphic in the vicinity of the points $z = z_k$ respectively, $v^{(k)}(z_k) = v_h^{(k)}(z_k) = 1$, $k = 0, 1$. The last condition fixes the normalization of the solutions. Here and below we suppose that the general situation for which the characteristic exponents are not integer is considered.

If we perform the analytical continuation of the fundamental pair $V^{(0)}(z)$ (along the real axis from point z_0 to point z_1) into the vicinity of the singular point z_1, then we can express the result of this procedure as a linear combination of the fundamental solutions $v_b^{(1)}(z)$, $v_h^{(1)}(z)$,

$$V^{(0)}(z) = \mathbf{K} V^{(1)}(z). \tag{4}$$

The corresponding matrix $\mathbf{K}$ is called the transition matrix [4]. It depends on the parameters of the CHE, $\mathbf{K} = \mathbf{K}(\alpha, \beta, \gamma, \delta, \epsilon)$.

The CHE has a set of elementary symmetries: s-homotopic symmetries [4] and the symmetry under reflection $z \to 1-z$. These symmetries transform the coefficients of the CHE, its form being not disturbed. The s-homotopic symmetries of the CHE include the symmetries under substitutions

$$v(z) = z^{\rho_0} u(z), \qquad v(z) = (1-z)^{\rho_1} w(z).$$

These symmetries produce the following relations between the different entries of the matrix $\mathbf{K}$ [9]:

$$\mathbf{K}_{21}(\alpha, \beta, \gamma, \delta, \epsilon) = \mathbf{K}_{11}(\alpha, \beta + 2(\gamma+1), -2-\gamma, \delta + \alpha(\gamma+1), \epsilon + (\beta+\gamma)(\gamma+1)), \tag{5}$$

$$\begin{aligned}\mathbf{K}_{12}(\alpha, \beta, \gamma, \delta, \epsilon) = \mathbf{K}_{11}(&\alpha, 2-2\alpha-2\gamma-\beta, \gamma, \delta + \alpha(1-\alpha-\beta-\gamma),\\ &\epsilon + (\alpha+\beta+\gamma-1)\gamma)),\end{aligned} \tag{6}$$

$$\mathbf{K}_{22}(\alpha, \beta, \gamma, \delta, \epsilon) = \mathbf{K}_{11}(\alpha, 4-2\alpha-\beta, -2-\gamma, \delta + \alpha(2-\alpha-\beta), \epsilon + 2 - \gamma\alpha - 2\alpha - \beta), \tag{7}$$

Hence, the transition matrix $\mathbf{K}$ can be determined exclusively by the single entry $\mathbf{K}_{11}(\alpha, \beta, \gamma, \delta, \epsilon)$, which we denote by $\mathbf{k}(\alpha, \beta, \gamma, \delta, \epsilon)$

The reflection symmetry of the CHE leads to a pair of identities for the function $\mathbf{k}(\alpha, \beta, \gamma, \delta, \epsilon)$ (see the details in [9]):

$$\begin{aligned}\mathbf{k}(\alpha, \beta, \gamma, \delta, \epsilon)&\mathbf{k}(-\alpha, \beta+2\alpha, -\alpha-\beta-\gamma, -\delta, \epsilon+\delta) + \mathbf{k}(\alpha, \beta+2(\gamma+1), -2-\gamma,\\ &\delta + \alpha(\gamma+1), \epsilon + (\beta+\gamma)(\gamma+1))\mathbf{k}(-\alpha, 2+\beta+2\alpha+2\gamma, -\alpha-\beta-\gamma,\\ &-\delta - \alpha(1+\gamma), \epsilon + \delta + (\alpha+\beta+\gamma)(1+\gamma)) = 1,\end{aligned} \tag{8}$$

$$\begin{aligned}\mathbf{k}(-\alpha, \beta+2\alpha, -\alpha-\beta-\gamma, -\delta, \epsilon+\delta)&\mathbf{k}(\alpha, 2-\beta-2\alpha-2\gamma, \gamma, \delta+\alpha(1-\alpha-\beta-\gamma),\\ &\epsilon + \gamma(\alpha+\beta+\gamma-1)) + \mathbf{k}(-\alpha, 2+\beta+2\alpha+2\gamma, -\alpha-\beta-\gamma, -\delta-\alpha(1+\gamma),\\ &\epsilon + \delta + (\alpha+\beta+\gamma)(1+\gamma))\mathbf{k}(\alpha, 4-\beta-2\alpha, -2-\gamma, \delta+\alpha(2-\alpha-\beta),\\ &\epsilon + 2 - \gamma\alpha - 2\alpha - \beta) = 0.\end{aligned} \tag{9}$$

The following simple result for the analytical continuation of solutions of the CHE (2) plays an important role in our procedure.

Lemma 1. *Let* $\mathbf{L}$ *be a loop beginning and ending in* z_0 *and surrounding the point* z_1 *(see figure 1). Under an analytic continuation along* $\mathbf{L}$ *the function* $v_b^{(0)}(z)$ *transforms as:*

$$v_b^{(0)}(z) \to v_b^{(0)}(z) + \zeta v_b^{(1)}(z), \tag{10}$$

where

$$\zeta = [\exp(2\pi \mathrm{i} \rho_1) - 1]\mathbf{k}(\alpha, \beta, \gamma, \delta, \epsilon). \tag{11}$$

Relations (10), (11) are simple consequences of the definitions [9]. Note that these relations include the transition matrix.

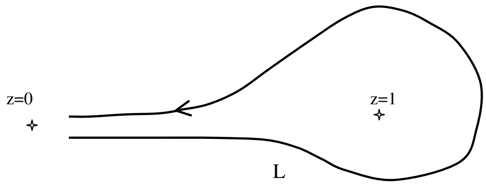

Figure 1. The path L of analytic connection of the function $v_b^{(0)}(z)$.

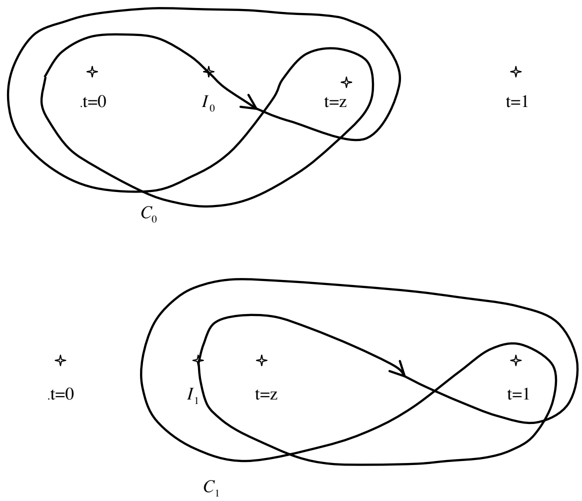

Figure 2. The contours of integration C_0 and C_1.

3. Integral symmetry

Let us recall some facts about Euler integral symmetry for the CHE obtained in [9].

Theorem 1. *Let C_0, C_1 be the double loops shown in figure 2 and $u_b^{(k)}(t)$, $k = 0, 1$, be the corresponding solutions of the confluent Heun equation*

$$\mathbf{R}_t[\alpha, 2+\beta-2\delta\alpha^{-1}, \gamma-1+\delta\alpha^{-1}, 2\alpha-\delta, \epsilon+(1-\delta\alpha^{-1})(\beta-\delta\alpha^{-1})]u(t) = 0. \tag{12}$$

Then, for some normalization constants q_k, we have:

$$\int_{C_k} (z-t)^{-\delta\alpha^{-1}} u_b^{(k)}(t)\,\mathrm{d}t = q_k v_b^{(k)}(z), \tag{13}$$

where $v_b^{(k)}(z)$ are described by the above solutions of the confluent Heun equation (2).

These facts can be easily proved with the help of integration by parts [9].

Remark 1. Let us fix the value of integrands in the following way. When all the parameters of equation (12) are real and variables z, t are real too, $0 < z < 1$, we fix the integrands real in the marked points on contours C_0, C_1 (these points belong to the real axis). As for the other values of variables, parameters and points on contours the corresponding values of the integrand can be evaluated by the analytical continuation. Then the normalization constants are

$$q_0 = 4\exp(\pi \mathrm{i}\gamma)\sin(\pi\delta\alpha^{-1})\sin(\pi(\gamma+\delta\alpha^{-1}))B(1+\gamma+\delta\alpha^{-1}, 1-\delta\alpha^{-1}), \tag{14}$$

$$q_1 = 4\exp(\pi \mathrm{i}(\delta\alpha^{-1} - \alpha - \beta - \gamma))\sin(\pi\delta\alpha^{-1})\sin(\pi(\delta\alpha^{-1} - \alpha - \beta - \gamma)) \times B(1 - \delta\alpha^{-1}, 1 + \delta\alpha^{-1} - \alpha - \beta - \gamma), \tag{15}$$

where $B(x, y)$ is the Euler beta-function [2].

We have two approaches for the description of the analytic continuation of $v_b^{(0)}(z)$ along the loop $\mathbf{L}$ (see figure 1). First, relations (10), (11) describe the result of the continuation in terms of function $\mathbf{k}(\alpha, \beta, \gamma, \delta, \epsilon)$. Relation (13) gives us the second tool: we can continue $v_b^{(0)}(z)$ along $\mathbf{L}$ with the corresponding transformation of the contour of integration C_0. This procedure expresses the result of the continuation in terms of equation (12) after some splitting of the deformed contour. Comparing it with (10), (11), we obtain the following relation for the transition matrix (see details in [9]):

$$\mathbf{k}(\alpha, \beta, \gamma, \delta, \epsilon) = \frac{\Gamma(\alpha + \beta + \gamma - 1)\Gamma(2 + \gamma)}{\Gamma(\alpha + \beta + \gamma - \delta\alpha^{-1})\Gamma(1 + \gamma + \delta\alpha^{-1})} \times \mathbf{k}[\alpha, 2 + \beta - 2\delta\alpha^{-1}, \gamma - 1 + \delta\alpha^{-1}, 2\alpha - \delta, \epsilon + (1 - \delta\alpha^{-1})(\beta - \delta\alpha^{-1})], \tag{16}$$

where $\Gamma(x)$ is the Euler gamma-function [2]. This relation reduces the correspondence between equations (2) and (12) from the level of solutions to the level of transition matrices. Moreover, the last formula gives us the possibility of calculating the transition matrix for the RCHE. Namely, if we suppose $\alpha \to 0$, then the left-hand side of (16) tends to the value $\mathbf{k}(0, \beta, \gamma, \delta, \epsilon)$ and we get

$$\mathbf{k}(0, \beta, \gamma, \delta, \epsilon) = \lim_{\alpha\to 0} \frac{\Gamma(\alpha + \beta + \gamma - 1)\Gamma(2 + \gamma)}{\Gamma(\alpha + \beta + \gamma - \delta\alpha^{-1})\Gamma(1 + \gamma + \delta\alpha^{-1})} \times \mathbf{k}[\alpha, 2 + \beta - 2\delta\alpha^{-1}, \gamma - 1 + \delta\alpha^{-1}, 2\alpha - \delta, \epsilon + (1 - \delta\alpha^{-1})(\beta - \delta\alpha^{-1})]. \tag{17}$$

So, the transition matrix of the second-order LODE with a large parameter α^{-1} is present on the right-hand side of (17). Applying the suitable asymptotic technique, one has to calculate the asymptotic expansion of the right-hand side of (16) and evaluate the limit value. This programme will be realized in the next section.

4. Asymptotic expansion

Here we will use a certain version of the comparison equation method [8] in order to construct the asymptotic of the transition matrix for equation (12), which is

$$u''(t) + \frac{\alpha t^2 + (a_0 + b_0 + 1)t - c_0}{t^2 - t}u'(t) + \frac{(2\alpha - \delta)t + \epsilon + a_0 b_0}{t^2 - t}u(t) = 0, \quad a_0 = 1 - \delta\alpha^{-1}, \quad b_0 = \beta - \delta\alpha^{-1}, \quad c_0 = 1 - \gamma - \delta\alpha^{-1}. \tag{18}$$

The leading asymptotic terms for the coefficients of this equation coincide with the coefficients of the hypergeometric equation (1) with coefficients $a = a_0, b = b_0, c = c_0$. The transition matrix $\mathbf{N}(a, b, c)$ for the hypergeometric equation (1) is known [2], and with the same assumptions for the indices we have:

$$\mathbf{N}_{11}(a, b, c) = \frac{\Gamma(2 - c)\Gamma(a + b - c)}{\Gamma(a + 1 - c)\Gamma(b + 1 - c)}. \tag{19}$$

So, we choose the hypergeometric equation (1) with suitable parameters $a(\alpha), b(\alpha), c(\alpha)$ as a comparison equation.

Before discussing the asymptotic procedure, we transform the equations (1), (18) into the more convenient form, namely, we omit the first derivatives. Consider the equation

$$u'' + Mu' + Nu = 0.$$

The substitution

$$u = AU, \qquad (\ln A)' = -M/2$$

yields

$$U'' + [N - M'/2 - M^2/4]U = 0.$$

For the hypergeometric equation (1),

$$M_1(s) = \frac{(a+b+1)s - c}{s^2 - s}, \tag{20}$$

we get

$$\begin{aligned} &\frac{\mathrm{d}^2 W(s)}{\mathrm{d}s^2} + R(a, b, c; s)W(s) = 0, \\ &R(a, b, c; s) = \frac{s^2[1 - (a-b)^2] + s[2c(a+b-1) - 4ab] + 1 - (1-c)^2}{4(s^2 - s)^2}. \end{aligned} \tag{21}$$

For equation (18),

$$M_2(t) = \frac{(a_0 + b_0 + 1)t - c_0 + \alpha t^2}{t^2 - t}, \tag{22}$$

and we have

$$\begin{aligned} &\frac{\mathrm{d}^2 U}{\mathrm{d}t^2} + Q(\alpha, t)U(t) = 0, \\ &Q(\alpha, t) = R(a_0, b_0, c_0; t) + \frac{2\epsilon(t^2 - t) + \delta t^2}{2(t^2 - t)^2} + O(\alpha) \end{aligned} \tag{23}$$

respectively.

Now let us shortly discuss the version of the comparison equation method used here. It is necessary to emphasize that this method in our situation is not well-grounded, so our considerations here are formal and heuristic. We look for a solution of equation (23) in the form

$$U(t) = [s'(t)]^{-1/2} W(s(t)), \tag{24}$$

where $W(s)$ is a solution of equation (21) and function $s(\alpha, t)$ describes the scaling of independent variable. The required values are the scaling function and parameters of the comparison equation (21) $a(\alpha)$, $b(\alpha)$, $c(\alpha)$. Taking into account (24), equation (23) yields an equation for the scaling function:

$$[s'(t)]^2 R(a, b, c; s(t)) - \frac{s'''(t)}{2s'(t)} + \frac{3(s''(t))^2}{4(s'(t))^2} = Q(\alpha, t). \tag{25}$$

In order to construct the solution of (23) with the prescribed behaviour in the singularity (for instance, at $t = 0$), we have to choose the respective solution of (21) and find the scaling function, which maps the singularity of (23) into the singularity of (21).

The asymptotic expansion of the parameters a, b, c is determined by the following conditions.

(i) In each asymptotic order of α the leading singularities in variable t (at $t = 0, 1$) on the left-hand and right-hand sides of (25) have to coincide;

(ii) The solution of the scaling equation (25) defined by the initial condition $s(\alpha, 0) = 0$ has to satisfy condition $s(\alpha, 1) = 1$.

These conditions determine the three parameters, that is the corresponding coefficients of the asymptotic expansion of $a(\alpha)$, $b(\alpha)$, $c(\alpha)$. The last condition guarantees that the scaling function $s(\alpha, t)$ maps the singularities of equation (23), $t = 0, 1$, into the singularities of equation (21), $s = 0, 1$.

Let us now describe the realization of the asymptotic procedure on the formal level. Namely, if we choose

$$a = 1 - \delta\alpha^{-1} - \kappa + \alpha, \qquad b = \beta - \delta\alpha^{-1} + \kappa, \qquad c = 1 - \gamma - \delta\alpha^{-1}, \tag{26}$$

where κ is determined by

$$\kappa^2 + \kappa(\beta - 1) + \epsilon + \delta/2 = 0, \tag{27}$$

we get: $R(a, b, c; t) = Q(\alpha, t) + O(\alpha)$. As $Q(\alpha, t) = O(\alpha^{-2})$, it follows from equation (25) that

$$s(\alpha, t) = t + O(\alpha^3). \tag{28}$$

Taking into account relations (24), (22), (20), and substituting (26) into (19), we obtain in the necessary asymptotic order:

$$\begin{aligned}&\mathbf{k}[\alpha, 2 + \beta - 2\delta\alpha^{-1}, \gamma - 1 + \delta\alpha^{-1}, 2\alpha - \delta, \epsilon + (1 - \delta\alpha^{-1})(\beta - \delta\alpha^{-1})]\\ &\qquad = \frac{\Gamma(\delta\alpha^{-1} + \gamma + 1)\Gamma(-\delta\alpha^{-1} + \alpha + \beta + \gamma)}{\Gamma(\beta + \gamma + \kappa)\Gamma(1 + \gamma - \kappa)}[1 + O(\alpha)].\end{aligned} \tag{29}$$

Note that the value of the right-hand side does not depend on the choice of solution of equation (27). Substituting the last relation into (17), we arrive at the final result:

$$\mathbf{k}(0, \beta, \gamma, \delta, \epsilon) = \frac{\Gamma(2 + \gamma)\Gamma(\beta + \gamma - 1)}{\Gamma(\beta + \gamma + \kappa)\Gamma(1 + \gamma - \kappa)}. \tag{30}$$

Let us consider some properties of this relation. First, if $\delta = 0$, RCHE reduces to the hypergeometric equation and (30) coincides with (19) after the transfer to the standard parameterizations (1) [2]. Then, relations (5)–(7) give the possibility of obtaining the values of the other entries of the transition matrix. A simple but cumbersome algebra shows that the relations (8), (9) are fulfilled (for $\alpha = 0$, respectively).

5. Numerical results

We have to emphasize that formulae (30) have no rigorous basis and it should be considered as an approximate result. To compare it with the results of the direct calculations of the value $\mathbf{k}(0, \beta, \gamma, \delta, \epsilon)$, we have performed the computer simulation. We calculated this value in accordance with its definition according to the following procedure. For $z_0 = 0.1$ we have assigned the approximate values of the functions

$$\begin{aligned}v_h^{(0)}(z_0) &= 1 + c_1 z_0 + c_2 z_0^2 + c_3 z_0^3,\\ v_b^{(0)}(z_0) &= z^{\rho_0}\big(1 + d_1 z_0 + d_2 z_0^2 + d_3 z_0^3\big),\end{aligned}$$

and theirs derivatives

$$\begin{aligned}\big[v_h^{(0)}\big]'(z_0) &= c_1 + 2c_2 z_0 + 3c_3 z_0^2 + 4c_4 z_0^3,\\ \big[v_b^{(0)}\big]'(z_0) &= z^{\rho_0 - 1}\big(\rho_0 + d_1(\rho_0 + 1)z_0 + d_2(\rho_0 + 2)z_0^2 + d_3(\rho_0 + 3)z_0^3\big),\end{aligned}$$

Here the coefficients $c_k, d_k, k = 1, 2, 3, 4$, can be derived from the recurrent relations. Note that here we have used only first four terms of the corresponding expansion. The approximate values of the functions $v_h^{(1)}(z_1)$, $v_b^{(1)}(z_1)$ at $z_1 = 0.9$ can be calculated in the similar way. These values have been set as the initial data for the solution of the equation (18) and for the calculation

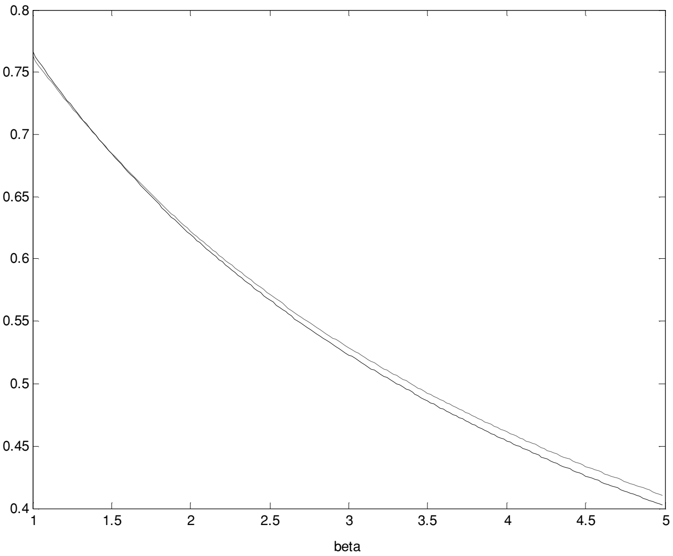

Figure 3. Graphics of the transition matrix element calculated by different methods. Gamma = 3.3, delta = 1.2, epsilon = −2.6, 1 < beta < 5.

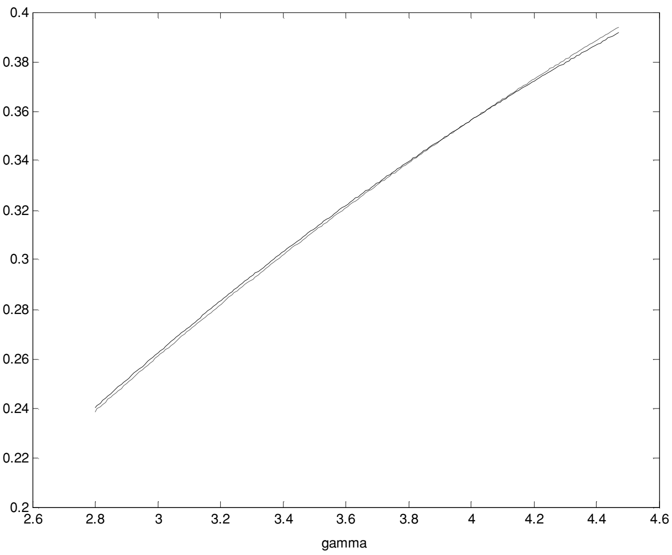

Figure 4. Graphics of the transition matrix element calculated by different methods. Beta = 1.5, delta = 2.2, epsilon = −6.3, 2.6 < gamma < 4.6.

of the values $v_h^{(0)}(z_t)$, $\left[v_h^{(0)}\right]'(z_t)$, $v_b^{(0)}(z_t)$, $\left[v_b^{(0)}\right]'(z_t)$, $v_h^{(1)}(z_t)$, $\left[v_h^{(1)}\right]'(z_t)$, $v_b^{(1)}(z_t)$, $\left[v_b^{(1)}\right]'(z_t)$ at $z_t = 0.5$, with the help of MatLab 6.0. Then the transition matrix is

$$\mathbf{K} = \begin{pmatrix} v_b^{(0)}(z_t) & \left[v_b^{(0)}\right]'(z_t) \\ v_h^{(0)}(z_t) & \left[v_h^{(0)}\right]'(z_t) \end{pmatrix} \begin{pmatrix} v_b^{(1)}(z_t) & \left[v_b^{(1)}\right]'(z_t) \\ v_h^{(1)}(z_t) & \left[v_h^{(1)}\right]'(z_t) \end{pmatrix}^{-1}. \tag{31}$$

The results of comparison of $\mathbf{K}_{11}$ and $\mathbf{k}(0, \beta, \gamma, \delta, \epsilon)$ are presented in figures 3–6. Note that the chosen approximate values for the initial data are not adequate for all values of parameters

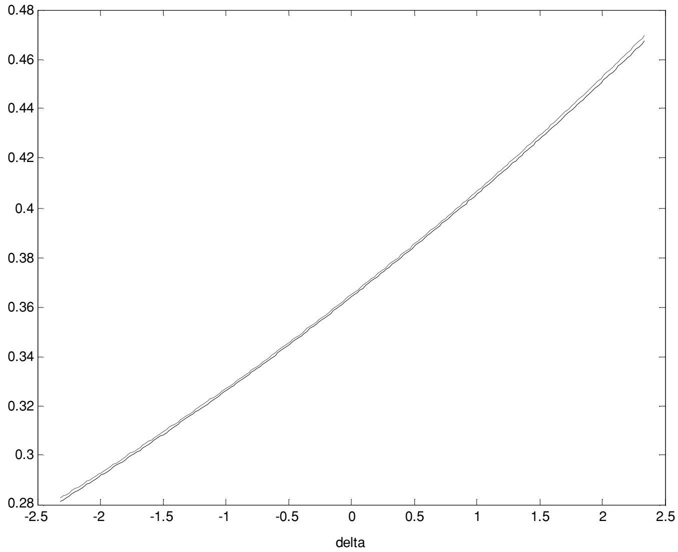

Figure 5. Graphics of the transition matrix element calculated by different methods. Beta = 1.5, gamma = 4.2, epsilon = −5.3, −2.5 < delta < 2.5.

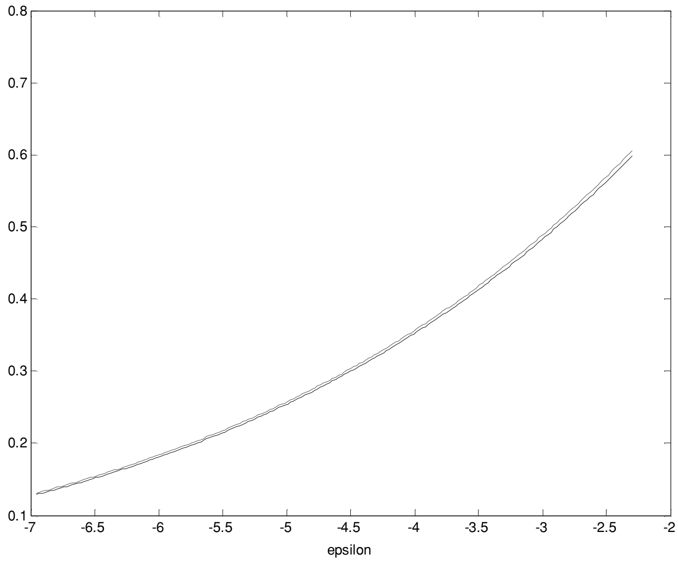

Figure 6. Graphics of the transition matrix element calculated by different methods. Beta = 2.5, gamma = 2.2, delta = 2.2, −7 < epsilon < −2.

$\beta, \gamma, \delta, \epsilon$. Particularly, the parameters β and γ with different signs are the parts of indices ρ_0, ρ_1. This fact leads to some singularities in the recurrent relations for the coefficients $c_k, d_k, k = 1, 2, 3, 4$ and theirs analogue for the solutions related to the point $z = 1$ at relatively large values of β and γ. So, we conclude that there is a good coincidence between functions $\mathbf{K}_{11}$ and $\mathbf{k}(0, \beta, \gamma, \delta, \epsilon)$.

6. Conclusion

Here we have presented the procedure for the calculation of the transition matrix (and, by the same token, of the monodromy matrices) which can be applied for confluent and reduced LODE. This procedure includes the derivation of the integral symmetry for more general equation, the reduction of this symmetry to the level of the transition matrix and the realization of the procedure of confluence or reduction in the last relation. The last step means a certain asymptotic problem related to the LODE of the same order. As an example of this procedure we have derived here the heuristic formulae for the transition matrix for the RCHE. Apparently, the analogous procedure can be realized for the confluent Heun equation, which is the most important Heun equation from the point of view of applications. We have compared our results with the results of direct computer simulations and have discovered the excellent agreement. It seems that the main obstacle to proving our procedure is the rigorous realization of the comparison equation method in the Cherry version for equation (18).

Note that the Laplace integral symmetry leads to the relations between the transition matrix of the given equation and Stokes parameters of the dual equation [9]. The dual equation for RCHE is the double confluent Heun equation [4].

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References

[1] Anosov D V and Bolibruch A A 1994 *The Riemann-Hilbert Problem* (Braunschweig: Vieweg)
[2] Bateman H and Erdelyi A 1953 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill)
[3] Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions* 9th ed (New York: Dover Publications)
[4] Slavyanov S Yu and Lay W 2000 *Special Functions* (Oxford: Oxford University Press)
[5] Kazakov A Ya 1998 *Teor. Matem. Fiz.* **116** 225 (in Russian)
[6] Cherry T M 1950 *Trans. Am. Math. Soc.* **68** 224
[7] Olver F W J 1974 *Asymptotics and Special Functions* (New York: Academic)
[8] Buldyrev V S and Slavyanov S Yu 1968 *Vestnik Leningradskogo Universiteta* **23** 70 (in Russian)
[9] Kazakov A Ya 2001 *Zap. nauchn. semin. POMI RAN* **264** 55 (in Russian)
Kazakov A Ya 2003 *J. Math. Sci.* **117** 3918 (Engl. Transl.)
[10] Fedoryuk M V 1993 *Asymptotic Methods* (Berlin: Springer)
[11] Hille E 1997 *Ordinary Differential Equations in the Complex Domain* (New York: Dover)